ON RIESZ AND RIEMANN SUMMABILITY

BY

DENNIS C. RUSSELL[1]

This paper investigates an inclusion relation between summability of a series of real or complex terms by Riesz typical means and by a generalised form of Riemann summability. We begin by defining the two summability methods.

Riesz' typical means. Let $\kappa \geqq 0$, $0 \leqq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$, and write

$$A_\lambda^\kappa(\omega) = \sum_{\lambda_\nu < \omega} (\omega - \lambda_\nu)^\kappa a_\nu \qquad \text{for } \omega > \lambda_0,$$

$$A_\lambda^\kappa(\omega) = 0 \qquad \text{for } \omega \leqq \lambda_0.$$

If $\omega^{-\kappa} A_\lambda^\kappa(\omega) \to s$ as $\omega \to \infty$ then we write

$$\sum_{n=0}^{\infty} a_n = s(R, \lambda_n, \kappa);$$

if $A_\lambda^\kappa(\omega) = O(\omega^\kappa)$ then $\sum a_n$ is bounded (R, λ_n, κ). In the case $\kappa = 0$ we note that

$$A_\lambda(\omega) \equiv A_\lambda^0(\omega) = \sum_{\lambda_\nu < \omega} a_\nu = a_0 + \cdots + a_n \equiv A_n$$

for $\lambda_n < \omega \leqq \lambda_{n+1}$ $(n = 0, 1, \cdots)$. It is well-known that $A_\lambda^\kappa(\omega)$ is absolutely continuous in any finite interval of values of ω, for $0 < \kappa \leqq 1$, and differentiable with continuous derivative if $\kappa > 1$; in fact,

$$\frac{d}{d\omega} A_\lambda^\kappa(\omega) = \kappa A_\lambda^{\kappa-1}(\omega) \ (\kappa > 1), \qquad \frac{d}{d\omega} A_\lambda^1(\omega) = A_\lambda(\omega) \ (\omega \neq \lambda_n). \tag{1}$$

As shown in Hardy and Riesz [9] or Chandresekharan and Minakshisundaram [5], we also have, for $\kappa \geqq 0$, $\rho > 0$,

$$A_\lambda^{\kappa+\rho}(\omega) = \frac{\Gamma(\kappa + \rho + 1)}{\Gamma(\kappa + 1)\Gamma(\rho)} \int_0^\omega (\omega - t)^{\rho-1} A_\lambda^\kappa(t) dt. \tag{2}$$

We shall employ the limitation theorem for Riesz means:

If $A_\lambda^\kappa(\omega) = O(\omega^\kappa)$, $\kappa \geqq 0$, *then, for* $r = 0, 1, \cdots, [\kappa]$,

Received by the editors August 17, 1961.

[1] This paper was written while the author was a Fellow at the 1961 Summer Research Institute of the Canadian Mathematical Congress.

$$(3) \qquad A_\lambda^r(\omega) = O(\omega^r \Lambda_n^{\kappa-r}),$$

where $\lambda_n < \omega \leqq \lambda_{n+1}$ *and* $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$.

The form of this theorem stated in [9, Theorem 22] and [5, Theorem 1.62] (we use O in place of o) is $A_\lambda^r(\omega) = O(\lambda_{n+1}^r \Lambda_n^{\kappa-r})$; the stronger form (3) is a special case of a result of Borwein [1, Lemma 2].

Finally, we need the "consistency theorem" for Riesz means:

$$(4) \qquad \textit{If } A_\lambda^\kappa(\omega) = O(\omega^\kappa), \quad \kappa \geqq 0, \quad \textit{then } A_\lambda^p(\omega) = O(\omega^p) \quad \textit{for } p \geqq \kappa.$$

Riemann summability. Let $\mu > 0$, $0 \leqq \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$,

$$f_\mu(x) = \left(\frac{\sin x}{x}\right)^\mu \ (x \neq 0), \qquad f_\mu(0) = 1;$$

if the series

$$\mathfrak{R}_\lambda^\mu(h) = \sum_{\nu=0}^\infty a_\nu f_\mu(\lambda_\nu h)$$

converges for each h in a deleted neighbourhood of the origin, and if $\mathfrak{R}_\lambda^\mu(h) \to s$ as $h \to 0$, then we write

$$\sum_{n=0}^\infty a_n = s \quad (\mathfrak{R}, \lambda_n, \mu).$$

The case where $\lambda_n = n$ and μ is a positive integer is usually known as Riemann summability. The more general definition above has been given by Burkill [2] for $\mu = 1, 2$, and by Burkill and Petersen [4] for μ rational with odd denominator (which ensures that $f_\mu(x)$ is real); alternatively, for any $\mu > 0$ we may define $(\sin x)^\mu = e^{i\mu\pi}(-\sin x)^\mu$ when $x > 0$, $\sin x < 0$, and $f_\mu(-x) = f_\mu(x)$. In fact, any definition is suitable for our purpose, which ensures that

$$\frac{d}{dx}(\sin x)^\mu = \mu(\sin x)^{\mu-1}\cos x, \qquad |(\sin x)^\mu| \leqq 1 \quad (\mu > 0),$$

$$|(\sin x)^\mu| \sim |x - n\pi|^\mu \quad (x \to n\pi);$$

and since $f_\mu(x)$ is an even function we may suppose throughout, in the definition of $(\mathfrak{R}, \lambda_n, \mu)$ summability, that $h > 0$.

Burkill [3] has shown that if $\lambda_0 = 0$, $0 < p \leqq \lambda_{n+1} - \lambda_n \leqq q$, and κ is a positive integer, then summability (R, λ_n, κ) implies summability $(\mathfrak{R}, \lambda_n, \mu)$ for $\mu > \kappa + 1$ (and μ rational with odd denominator). Burkill and Petersen [4] have proved this for $\kappa = 1$, remarking that from the point of view of applications (for instance, to the theory of almost periodic functions—see, for example, [2] and [11]) it would be desirable to proceed from a nonintegral Riesz mean to an integral Riemann mean. The present paper furnishes such a result, which also contains the theorem referred to above; we prove, more generally, the following

THEOREM. *If* $\sum_{n=0}^{\infty} a_n = s(R, \lambda_n, \kappa)$, $\kappa \geqq 0$, *and if* $\sum_{n=1}^{\infty} \Lambda_n^{\kappa} \lambda_n^{-\mu}$ *converges, where* $\Lambda_n = \lambda_{n+1}/(\lambda_{n+1} - \lambda_n)$ *and* $\mu > \kappa + 1$, *then* $\sum_{n=0}^{\infty} a_n = s(\mathfrak{R}, \lambda_n, \mu)$.

In the special case $\lambda_n = n$, (R, λ_n, κ) is equivalent to Cesàro summability (C, κ), and $(\mathfrak{R}, \lambda_n, \mu)$ becomes ordinary Riemann summability, which will be denoted by $(\mathfrak{R}, \mu)$; if, in addition, μ is a positive integer greater than 1, we obtain a result of Verblunsky [12] that $(C, \kappa) \subseteq (\mathfrak{R}, \mu)$ for $0 \leqq \kappa < \mu - 1$, $\mu = 2, 3, \cdots$; Hardy and Littlewood [7; 8] had proved earlier that $(C, \kappa) \subseteq (\mathfrak{R}, 1)$ for $-1 \leqq \kappa < 0$. Kuttner [10] has proved that $(\mathfrak{R}, \mu) \subseteq (C, \mu + \delta)$ for $\delta > 0$, $\mu = 1, 2$, and that the result is false for $\mu = 3$; and he has shown that $(\mathfrak{R}, \mu) = (\mathfrak{R}, n, \mu) \subseteq (R, \log n, \mu)$ for $\mu = 1, 2$. See also Hardy [6, Appendix III].

Some lemmas are needed. We remark that in general throughout this paper K will denote a positive quantity independent of the particular variables under consideration, and not necessarily the same at each occurrence; thus, for example, in the first lemma the constants K may depend on μ or p, but are independent of x or n.

LEMMA 1. *Let p be a non-negative integer, and define* $f_0(x) \equiv 1$.

(a) *For any* $\mu \geqq p$, $f_\mu^{(p)}(x)$ *is continuous everywhere, and*

$$(5)(^2) \quad \left| f_\mu^{(r)}(x) \right| \leqq K \ (0 < x < 1), \ \left| f_\mu^{(r)}(x) \right| \leqq K x^{-\mu} \ (x \geqq 1), \ r = 0, 1, \cdots, p.$$

(b) *If* $\mu > p$ *then* $f_\mu^{(r)}(n\pi) = 0$ $(n = 1, 2, \cdots; \ r = 0, 1, \cdots, p)$. *Also* $f_\mu^{(p+1)}(x)$ *is continuous in* $(n-1)\pi < x < n\pi$ $(n = 1, 2, \cdots)$ *and, in each such interval, satisfies the inequality*

$$(6) \qquad \left| f_\mu^{(p+1)}(x) \right| \leqq K n^{-\mu} \{ (n\pi - x)^{\mu - p - 1} + [x - (n-1)\pi]^{\mu - p - 1} \}.$$

Proof. We first note that, for each non-negative integer s,

$$(7) \qquad \left| f_1^{(s)}(x) \right| \leqq K \ (0 < x < 1), \qquad \left| f_1^{(s)}(x) \right| \leqq K x^{-1} \ (x \geqq 1);$$

the first of these inequalities is an immediate consequence of the fact that $f_1(x)$ has a power series expansion with infinite radius of convergence, while the second follows from the formula

$$f_1^{(s)}(x) = \sum_{k=0}^{s} \binom{s}{k} (-1)^k k! x^{-k-1} \sin\left[x + \frac{1}{2}(s - k)\pi \right].$$

It is clear that $f_\mu(x)$ is differentiable as often as we please, except perhaps at $x = \pm\pi, \pm 2\pi, \cdots$; also $f'_{\mu+1}(x) = (\mu+1) f_\mu(x) f_1'(x)$, and on differentiating p times this gives

(2) This inequality is also given (for μ rational with odd denoninator) in [3, Lemma 2].

$$f_{\mu+1}^{(p+1)}(x) = (\mu + 1) \sum_{r=0}^{p} \binom{p}{r} f_\mu^{(r)}(x) f_1^{(p-r+1)}(x), \tag{8}$$

which enables us to proceed by induction on p. We shall merely verify the inequalities (5) and (6).

(a) Suppose that, for some fixed non-negative integer p and for any $\mu \geqq p$, (5) holds; then since $\mu \geqq p$ implies $\mu+1 \geqq p$, (5) also holds with $\mu+1$ in place of μ (and $r=0, 1, \cdots, p$). Further, (8) shows, by (7) and the inductive hypothesis, that $f_{\mu+1}^{(p+1)}(x)$ is bounded in (0, 1) and is $O(x^{-\mu-1})$ as $x \to \infty$. Since (5) may be verified directly from the definition of $f_\mu(x)$ in the case $p=0$, it follows that (5) is true for any non-negative integer p and any $\mu \geqq p$.

(b) If $\mu \geqq p+1$ then (6) is equivalent to $|f_\mu^{(p+1)}(x)| \leqq Kn^{-\mu}$ for $0 \leqq (n-1)\pi < x < n\pi$, which has already been proved in part (a) of the lemma. Suppose, therefore, that for some fixed non-negative integer p and $0 < |n\pi - x| \leqq \pi/2$ $(n=0, 1, \cdots)$,

$$|f_\mu^{(p+1)}(x)| \leqq K(n+1)^{-\mu} |n\pi - x|^{\mu-p-1} \qquad \text{for } p < \mu < p+1; \tag{9}$$

in addition, we already know from (5) and (7) that

$$\begin{aligned} |f_\mu^{(r)}(x)| &\leqq K(n+1)^{-\mu} \quad (r = 0, 1, \cdots, p), \\ |f_1^{(s)}(x)| &\leqq K(n+1)^{-1} \quad (s = 0, 1, \cdots). \end{aligned} \tag{10}$$

Now use (8) with $p+1$ in place of p, together with (9) and (10), and we get

$$|f_{\mu+1}^{(p+2)}(x)| \leqq K(n+1)^{-\mu-1} |n\pi - x|^{\mu-p-1} + K(n+1)^{-\mu-1};$$

or, writing ν for $\mu+1$,

$$|f_\nu^{(p+2)}(x)| \leqq K(n+1)^{-\nu} |n\pi - x|^{\nu-p-2} \qquad \text{for } p+1 < \nu < p+2.$$

Since we may verify (9) directly for $p=0$, (9) therefore follows, by induction, for any non-negative integer p; and by combining the results for the two halves of the interval $(n-1)\pi < x < n\pi$, we obtain (6).

Defining $A_{n+1}(A_{-1}=0)$ and $A_\lambda(\tau)$ as before, we now prove

LEMMA 2. *If* $\mu \geqq 1$, $\lambda_n < \Omega \leqq \lambda_{n+1}$ $(n=0, 1, \cdots)$, *then*

$$\sum_{\nu=0}^{n} a_\nu f_\mu(\lambda_\nu h) = A_\lambda(\Omega) f_\mu(\Omega h) - h \int_0^\Omega f_\mu'(\tau h) A_\lambda(\tau) d\tau. \tag{11}$$

Proof. Since $f_\mu'(x)$ is continuous for any x, when $\mu \geqq 1$, and $A_\lambda(\tau) = A_\nu$ for $\lambda_\nu < \tau \leqq \lambda_{\nu+1}$ we have, for $\lambda_n < \Omega \leqq \lambda_{n+1}$,

$$h\int_0^\Omega f'_\mu(\tau h)A_\lambda(\tau)d\tau = h\left\{\sum_{\nu=0}^{n-1}\int_{\lambda_\nu}^{\lambda_\nu+1} + \int_{\lambda_n}^{\Omega}\right\} f'_\mu(\tau h)A_\lambda(\tau)d\tau$$

$$= h\sum_{\nu=0}^{n-1} A_\nu\left[\frac{1}{h}f_\mu(\tau h)\right]_{\lambda_\nu}^{\lambda_\nu+1} + hA_n\left[\frac{1}{h}f_\mu(\tau h)\right]_{\lambda_n}^{\Omega}$$

$$= A_nf_\mu(\Omega h) - \sum_{\nu=0}^{n}(A_\nu - A_{\nu-1})f_\mu(\lambda_\nu h),$$

by partial summation; and this gives (11).

Now to obtain $\mathfrak{R}_\lambda^\mu(h)$ we must let $n\to\infty$ in (11); the following lemma gives sufficient conditions for the existence of $\mathfrak{R}_\lambda^\mu(h)$.

LEMMA 3. *If $\sum a_n$ is bounded (or summable) (R, λ_n, κ), $\kappa\geqq 0$, and if $\sum\Lambda_n^\kappa\lambda_n^{-\mu}$ converges, then $\sum a_nf_\mu(\lambda_nh)$ converges (absolutely) for each fixed $h>0$.*

Proof. If $\sum a_n$ is bounded (R, λ_n, κ) then by (3) (with $r=0$), $A_n=O(\Lambda_n^\kappa)$; moreover, for any fixed $h>0$, $f_\mu(\lambda_nh)=O(\lambda_n^{-\mu})$ as $n\to\infty$. Hence

$$a_nf_\mu(\lambda_nh) = (A_n - A_{n-1})f_\mu(\lambda_nh)$$

$$= \left\{O(\Lambda_n^\kappa) + O(\Lambda_{n-1}^\kappa)\right\}O(\lambda_n^{-\mu})$$

$$= O(\Lambda_n^\kappa\lambda_n^{-\mu}) + O(\Lambda_{n-1}^\kappa\lambda_{n-1}^{-\mu}),$$

and the lemma follows.

LEMMA 4. *Let p be a positive integer, $0\leqq\sigma<1$, $\mu>p$, and*

$$I(\alpha) = \int_\alpha^\infty (x-\alpha)^{-\sigma}df_\mu^{(p)}(x).$$

Then

$$|I(\alpha)| \leqq Kn^{-\mu}\{(n\pi-\alpha)^{-\sigma} + [\alpha-(n-1)\pi]^{\mu-p-1}\}$$

when $(n-1)\pi<\alpha<n\pi$, $n=1, 2, \cdots$.

Proof. Let $(n-1)\pi<\alpha<n\pi$; then

$$I(\alpha) = \left\{\int_\alpha^{n\pi} + \int_{n\pi}^\infty\right\}(x-\alpha)^{-\sigma}df_\mu^{(p)}(x) \equiv J_1 + J_2, \text{ say.}$$

Since, by Lemma 1, $f_\mu^{(p)}(n\pi)=0$ and $|f_\mu^{(p)}(x)|\leqq Kx^{-\mu}(x\geqq 1)$, we have, for $\sigma\geqq 0$, $\mu>p$, on integrating by parts,

$$|J_2| = \left|\sigma\int_{n\pi}^\infty (x-\alpha)^{-\sigma-1}f_\mu^{(p)}(x)dx\right| \tag{12}$$

$$\leqq Kn^{-\mu}(n\pi-\alpha)^{-\sigma}.$$

Noting that $0 \leqq \sigma < 1$, $\mu > p$, $0 < n\pi - \alpha < \pi$, we now use (6), together with the formula

$$\int_a^b (x-a)^{q-1}(b-x)^{r-1}dx = (b-a)^{q+r-1}\mathrm{B}(q, r) \quad (q, r > 0);$$

then

$$
\begin{aligned}
|J_1| &= \left| \int_\alpha^{n\pi} (x-\alpha)^{-\sigma} f_\mu^{(p+1)}(x)dx \right| \\
&\leqq Kn^{-\mu}\left\{ \int_\alpha^{n\pi} (x-\alpha)^{-\sigma}(n\pi - x)^{\mu-p-1}dx \right. \\
&\qquad\qquad \left. + \int_\alpha^{n\pi} (x-\alpha)^{-\sigma}[x-(n-1)\pi]^{\mu-p-1}dx \right\} \\
&\leqq Kn^{-\mu}\left\{ (n\pi-\alpha)^{\mu-p-\sigma} + (n\pi-\alpha)^{1-\sigma}[\pi^{\mu-p-1} + (\alpha - n\pi - \pi)^{\mu-p-1}] \right\} \\
&\leqq Kn^{-\mu}\left\{ (n\pi-\alpha)^{-\sigma} + [\alpha - (n-1)\pi]^{\mu-p-1} \right\}.
\end{aligned} \tag{13}
$$

Since $|I(\alpha)| \leqq |J_1| + |J_2|$, the lemma now follows from (12) and (13).

Proof of the Theorem. We may suppose that $\kappa = \sigma + p - 1$, where $0 \leqq \sigma < 1$ and p is a positive integer. By (1) and Lemma 2 we have, for $\mu > p$ and $\lambda_n < \Omega \leqq \lambda_{n+1}$,

$$
\begin{aligned}
\sum_{\nu=0}^{n} a_\nu f_\mu(\lambda_\nu h) &= A_\lambda(\Omega) f_\mu(\Omega h) - h\int_0^\Omega f'_\mu(\tau h)dA_\lambda^1(\tau) \\
&= \sum_{r=0}^{p} \frac{(-h)^r}{r!} A_\lambda^r(\Omega) f_\mu^{(r)}(\Omega h) + \frac{(-1)^{p+1}h^p}{p!}\int_0^\Omega A_\lambda^p(\tau) df_\mu^{(p)}(\tau h),
\end{aligned} \tag{14}
$$

after p integrations by parts ($A_\lambda^r(0) = 0$). Using (2) with $\rho = 1 - \sigma$, $\kappa = \sigma + p - 1$, and writing $C = \{\Gamma(\sigma + p)\Gamma(1-\sigma)\}^{-1}$,

$$
\begin{aligned}
\frac{1}{p!}\int_0^\Omega A_\lambda^p(\tau) df_\mu^{(p)}(\tau h) &= C\int_0^\Omega df_\mu^{(p)}(\tau h) \int_0^\tau (\tau - t)^{-\sigma} A_\lambda^{\sigma+p-1}(t)dt \\
&= C\int_0^\Omega A_\lambda^{\sigma+p-1}(t)dt \int_t^\Omega (\tau - t)^{-\sigma} df_\mu^{(p)}(\tau h) \\
&= C\int_0^\Omega A_\lambda^{\sigma+p-1}(t)dt \left\{ \int_t^\infty - \int_\Omega^\infty \right\} (\tau - t)^{-\sigma} df_\mu^{(p)}(\tau h) \\
&\equiv I_1 - I_2, \text{ say.}
\end{aligned} \tag{15}
$$

For each fixed $h > 0$, $\sigma \geqq 0$, $\mu > p \geqq 1$, for $t < \Omega$, and for all $\Omega \geqq h^{-1}$ we have, on integrating by parts and using $|f_\mu^{(p)}(x)| \leqq Kx^{-\mu} (x \geqq 1)$,

$$\left| \int_\Omega^\infty (\tau - t)^{-\sigma} df_\mu^{(p)}(\tau h) \right| \leqq K\Omega^{-\mu}(\Omega - t)^{-\sigma},$$

where K is independent of Ω and t. Since, by hypothesis, $A_\lambda^{\sigma+p-1}(t)=O(t^{\sigma+p-1})$, it then follows that, as $\Omega\to\infty$,

$$|I_2| \leqq K\int_0^\Omega t^{\sigma+p-1}\Omega^{-\mu}(\Omega-t)^{-\sigma}dt \leqq K\Omega^{p-\mu}\to 0.$$

We now observe that, for $r=0, 1, \cdots, p-1$, (3) and (5) give

$$\begin{aligned} A_\lambda^r(\Omega)f_\mu^{(r)}(\Omega h) &= O\{\Omega^r\Lambda_n^{\kappa-r}\Omega^{-\mu}\} \\ &= O\{\Lambda_n^\kappa\lambda_n^{-\mu}(\lambda_n/\Lambda_n)^r\} \\ &= O\{\Lambda_n^\kappa\lambda_n^{-\mu}\}O\{1+(\lambda_n/\Lambda_n)^\kappa\} \\ &= O\{\Lambda_n^\kappa\lambda_n^{-\mu}\}+O\{\lambda_n^{\kappa-\mu}\} \\ &= o(1)+o(1), \end{aligned}$$

since $\mu>p>\kappa\geqq r$ and $\sum\Lambda_n^\kappa\lambda_n^{-\mu}$ converges; while, by (4) and (5),

$$A_\lambda^p(\Omega)f_\mu^{(p)}(\Omega h) = O\{\Omega^p\Omega^{-\mu}\} = o(1).$$

Thus the series on the right of (14) tends to zero as $\Omega\to\infty$, while (by Lemma 3) the series on the left tends to a limit $\Re_\lambda^\mu(h)$. Hence the integral on the right of (14) tends to a limit; then, since $I_2\to 0$, we may let $\Omega\to\infty$ in (15) and substitute the result into (14) to give, for $h>0$,

$$\Re_\lambda^\mu(h) = C(-1)^{p+1}\int_0^\infty \phi(h,t)t^{-\kappa}A_\lambda^\kappa(t)dt, \tag{16}$$

where

$$\phi(h,t) = h^p t^\kappa\int_t^\infty (\tau-t)^{-\sigma}df_\mu^{(p)}(\tau h). \tag{17}$$

The theorem will then follow if we can show that $t^{-\kappa}A_\lambda^\kappa(t)\to s$ as $t\to\infty$ implies $\Re_\lambda^\mu(h)\to s$ as $h\to 0+$; by Hardy [6, Theorem 6], sufficient conditions for this are:

$$\int_0^\infty |\phi(h,t)|\,dt \leqq M \text{ independently of } h>0, \tag{18}$$

$$\lim_{h\to 0+}\int_0^T |\phi(h,t)|\,dt = 0 \text{ for every finite } T>0, \tag{19}$$

$$\lim_{h\to 0+} C(-1)^{p+1}\int_0^\infty \phi(h,t)dt = 1. \tag{20}$$

For (20) we can apply (16) to sequences $\{\lambda_n\}$, $\{a_n\}$ satisfying $\lambda_0=0$, $a_0=1$, $a_n=0$ $(n\geqq 1)$ to obtain at once

$$1 = C(-1)^{p+1}\int_0^\infty \phi(h, t)dt$$

for any $h>0$, since for the sequences in question

$$A_\lambda^\kappa(t) = t^\kappa \ (t > 0), \qquad \mathfrak{R}_\lambda^\mu(h) = 1.$$

Now the substitution $x=\tau h$, $\alpha=th$ in (17) gives

$$\int_0^T |\phi(h, t)|\, dt = \int_0^{Th} \alpha^\kappa |I(\alpha)|\, d\alpha,$$

where

$$I(\alpha) = \int_\alpha^\infty (x-\alpha)^{-\sigma} df_\mu^{(p)}(x).$$

Thus both (18) and (19) will follow if we can show that

$$\int_0^\infty \alpha^\kappa |I(\alpha)|\, d\alpha < \infty. \tag{21}$$

But by Lemma 4,

$$\begin{aligned}\int_0^\infty \alpha^\kappa |I(\alpha)|\, d\alpha &= \sum_{n=1}^\infty \int_{(n-1)\pi}^{n\pi} \alpha^\kappa |I(\alpha)|\, d\alpha \\ &\leqq K\sum_{n=1}^\infty n^\kappa \int_{(n-1)\pi}^{n\pi} n^{-\mu}\{(n\pi-\alpha)^{-\sigma} + [\alpha-(n-1)\pi]^{\mu-p-1}\} d\alpha \\ &\leqq K\sum_{n=1}^\infty n^{\kappa-\mu} \text{ since } \sigma < 1 \text{ and } \mu > p \\ &< \infty \quad \text{when} \quad \mu > \kappa + 1,\end{aligned}$$

so that (21) holds and the proof is complete.

References

1. D. Borwein, *On the abcissae of summability of a Dirichlet series*, J. London Math. Soc. **30** (1955), 68–71.

2. H. Burkill, *Cesàro-Perron almost periodic functions*, Proc. London Math. Soc. (3) **2** (1952), 150–174.

3. ———, *On Riesz and Riemann summability*, Proc. Cambridge Philos. Soc. **57** (1961), 55–60.

4. H. Burkill and G. M. Petersen, *A relation between Riesz and Riemann summability*, Proc. Amer. Math. Soc. **12** (1961), 453–456.

5. K. Chandresekharan and S. Minakshisundaram, *Typical means*, University Press, Oxford, 1952.

6. G. H. Hardy, *Divergent series*, University Press, Oxford, 1949.

7. G. H. Hardy and J. E. Littlewood, *Two theorems concerning Fourier series*, J. London Math. Soc. **1** (1926), 19–25.

8. ———, *On Young's criterion for Fourier series*, Proc. London Math. Soc. (2) **28** (1928), 301–311.

9. G. H. Hardy and M. Riesz, *The general theory of Dirichlet series*, Cambridge Tract No. 18 (1915, reprinted 1952).

10. B. Kuttner, *The relation between Riemann and Cesàro summability*, Proc. London Math. Soc. (2) **38** (1935), 273–283.

11. R. G. Uttley, *Almost periodicity and convergent trigonometric series*, Trans. Amer. Math. Soc. **99** (1961), 414–424.

12. S. Verblunsky, *The relation between Riemann's method of summation and Cesàro's*, Proc. Cambridge Philos. Soc. **26** (1929), 34–42.

Mount Allison University,
Sackville, New Brunswick, Canada